A Short Account of the History of Structural Dynamics between the Nineteenth and Twentieth Centuries

Massimo Corradi

THE FOUNDATION OF DYNAMICS: A BRIEF INTRODUCTION

What a difficult task it is to give a short account of the history of the structural dynamics! It's a complex and diversified history, coming from different sources and with influences from interstitial (or interface) areas, moving between different branches of engineering, crossing them transversally and deriving from each a new impulse for development. Moreover, dynamics is a field emerging somewhere between mathematics, physics and mechanics. Also, dynamics has evolved into more disciplines: applied mathematics, theoretical mechanics, and experimental physics. The oldest of these disciplines is applied dynamics, which originally was regarded as a branch of natural philosophy or physics related to natural phenomena, and its origin goes back to Galileo Galilei (1564-1642), at least. Nevertheless, dynamics is very old discipline. The history of dynamics started with the studies of Aristotle (384-322 B.C.). Aristotle's *Physics* was the first step on a long journey. Aristotle thought deeply about two fundamental questions debated by Parmenides (Fifth century B.C.) and Heracleitus (c.550-480 B.C.), on the reality and mechanisms of dynamics. What is change? Is it real? Why do things change? Aristotle realised that we understand change through duality. He modelled physical change with 'matter' and 'form'. Going beyond physics, he modelled metaphysical change with 'potency' and 'act'. Zeno of Elea (490-430 B.C.) developed many arguments showing that motion is impossible. Zeno's paradoxes support the position of Parmenides, who felt that reality was eternal and motion an illusion. (The invention of the calculus by Newton and Leibniz would make the logical treatment of motion, continuity and infinity live issues in mathematics).

Then, dynamics resumed its journey with Aristarchus of Samos (310-230 B.C.), who proposed the Heliocentric theory, Hipparchus of Rhodes (190-120 B.C.), who measured the angular height of the star Alpha Virgins above the ecliptic and compared his measures with Babylonian observations. Hipparchus deduced that the Earth's axis precede at 47 arc-seconds per year and also made detailed observations of the moon, and estimated the earth-moon distance with a good accuracy. Ptolomey (c.100-178A.D.) knocks heliocentricity on the head because it violates Aristotle's ideas. He then wrote a detailed mathematical theory of the motion of the sun, moon, and planets. In the Middle Ages Thomas Aquinas (1222-1274) combined Aristotelian metaphysics with Christian belief to produce the most influential work on the nature of God even written, his *Summa Theologiae* (Aquinas 1509). In the Renaissance, Galileo was one of the first to deal thoroughly with the concept of acceleration and he founded dynamics as a branch of natural philosophy. The close interplay of theory and experiment, characteristic of this subject, was founded by Italian scientists. Galileo said mathematics is the means to decipher the book of nature. Mathematics seeks to discern the outlines of all possible abstract structure. This pure mathematics may be applied to every sort of concrete problem. Consequently the history of mathematics is as old as the history of philosophy, and mathematical discoveries have often influenced philosophy. But,

> "the main kinematical properties of uniformly accelerated motions, still attributed to Galileo by the physics texts, were discovered and proved by scholars of Merton College – William Heytesbury, Richard Swineshead, and John of Dumbleton – between 1328 and 1350. Their work distinguished *kinematics*, the geometry of motion, from *dynamics*, the theory of the causes of motion"
>
> (Truesdell 1968, p. 30).

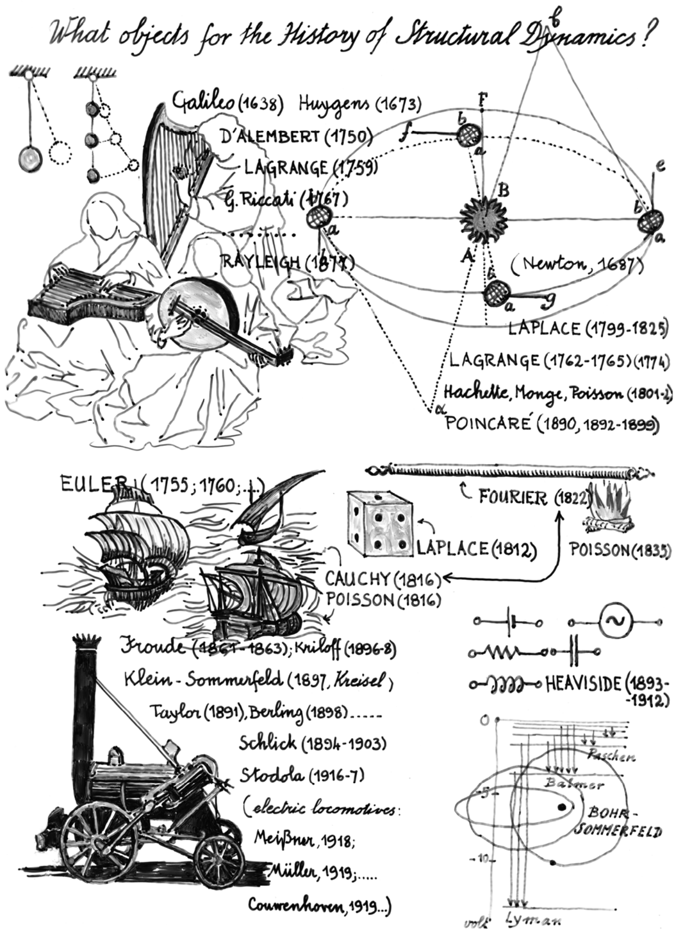

Figure 1. "What objects for the History of Structural Dynamics?" (Edoardo Benvenuto)

Tycho Brahe (1546-1601) applied dynamics to the study of celestial mechanics. Johannes Kepler (1629-1695) was the outstanding and original exponent of applied dynamics. Kepler made use of extensive interaction between theory and observation to understand the planetary motion. Christiaan Huygens (1629-1695) in 1656 patented the first pendulum clock and applied it to longitude determination. Isaac Newton (1642-1727) founded mathematical dynamics, as well as the calculus on which it is based. Applications and experiments were basic to his ideas, which were dominated by the doctrine of determinism, but his methods were geometrical. The concept of calculus, mathematical dynamics, and their implications for natural philosophy, occurred independently to Gottfried Wilhelm Leibniz (1646-1716), but his methods were more symbolic than geometrical. John Harrison (1693-1776) made important contributions to the *mécanique céleste* and his work fixed the new topic of dynamics. Otherwise, mathematical dynamics began with Newton and has become a large and active branch of pure mathematics.

In the eighteenth century Leonhard Euler (1707-1783) developed the technique of analysis that were to dominate mathematical dynamics throughout its classical period. Louis Lagrange (1736-1813), developed the analytical method to extremes, and boasted that his definitive text on the subject contained not a single illustration, Daniel Bernoulli (1700-1782) applied dynamics to Hydrodynamics, established the most important steps of dynamics. Over a period of 41 years, 1766-1817, Pierre Simon Laplace (1749-1827) took important steps and put the (gamma-1)/gamma into the speed of sound. His career peaked in 1812 when his work on probability was published. In 1762, Lagrange writes his "Method of Variations" and in 1766 he won prizes for his work on Moon, Jupiter, 3-body problem and comets. He also wrote his *Méchanique Analytique*, which contained no diagrams and in 1787, the publication of *Mécanique Céleste* commences.

Finally, in the nineteenth century Henry Poincaré (1854-1912) became the first to consider the possibility of chaos in a deterministic system, in his work on planetary orbits. Little interest was shown in this work until the modern study of chaotic dynamics began in 1963. But, since Poincaré the newer methods of topology and geometry have dominated the field of theoretical dynamics. Marius Sophus Lie (1842-1899), combining the ideas of symmetry and dynamics built the foundations for a far-reaching extension of dynamics, the theory of groups of transformations. John William Strutt, Baron Rayleigh (1842–1919) dwelled at length on acoustical physics. In this work he revived the experimental tradition of Galileo in dynamics, laying the foundation for the theory of non-linear oscillations. His text on acoustics, published in 1877, remains to this day the best account of this subject.

STRUCTURAL DYNAMICS: SOME SKETCHES

Over the last two centuries, the growth of structural dynamics was stimulated by theoretical investigations and computational methods arising from other contexts: from the theory of sound to the *mécanique céleste*; from the *théorie de la chaleur* to electromagnetism, and from fluid dynamics to atomic physics. In the case of dynamic effects on structures, a strenuous and in-depth discussion came in several different fields, in mechanical engineering (design and construction of machineries, control of vibration, etc.), in naval, aeronautics and also in civil engineering. For example, the problems of bridges under the action of heavy loads moving at speed, industrial buildings subjected to dynamic actions, constructions in earthquake zones, wind effects on tall buildings and particular structures, etc.

From the first half of the twentieth century, the dynamic analysis of structures gradually began to be articulated and compounded, taking shape as a separate discipline arising from the theory of structures and the strength of materials. However, its most perspicuous applications continued to be improved by groups of scientists, who were directly involved with specific problems of concern to other technological sectors.

A further remark concerns the mathematical language adopted by the discipline of structural dynamics, which over time gave consistency and consonance to its theoretical shape and made it possible to simplify and to improve its demonstrative arguments and methods of calculation. Sometimes a simple change of notation was sufficient to give a new impulse to mathematical interpretations about results which had been known ever since Euler, Johann Bernoulli (1667–1748), Jean-Baptiste Le Rond d'Alembert (1717–1783) and Lagrange, but that were still not recognized for their full significance. As Laplace said: "A well devised notation is sometimes half the battle in mathematics." From a historical perspective, especially in our century, the growth of structural dynamics is an important example of how some of the most important results were produced by the development of mathematics in the continuous transformation of its language and, even more so, in the evolution of its ideas.

Among the promoters and contributors to structural dynamics, we cannot count just physicists and engineers, but also mathematicians devoted to theoretical studies: from Hermann Grassmann (1809-1877) to Arthur Cayley (1821-1895), from Georg Bernhard Riemann (1826-1866) to David Hilbert (1862-1943) and then to John von Neumann (1903-1957).

The linguistic metamorphosis within structural dynamics over the last fifty years has been so deep that, if a student today takes a recently published textbook, he is likely to believe that it would not be possible to write the fundamental equations for structural systems and to look for their solution, without the many elements of contemporary mathematics, which have been developed in the twentieth century, such as abstract algebra, linear spaces, spectral theory of operators, functional analysis, etc. Studying history we notice that these events ordered how the solution of equations - originally written in different form - was arrived at without laborious calcula, when, in the great book of the history of mathematics, the required chapters still had to be written.

It was thanks to this earlier, practical work, concerning actual problems that the discovery of new fields of research - which mark contemporary mathematics - was possible. In other words, the mathematical formalization of structural methods grew, as Georg Wilhelm Friedrich Hegel (1770-1788) observed about philosophy, like Minerva's noctule, "it flied off towards the evening". Karl Friedrich Gauss (1777-1855) made a similar observation when writing a letter to Heinrich Christian Schumacher (1780-1850) concerning his *Barycentrischer Calcul*, in 1843:

> It's widely held that by these new methods, you do not obtain anything that you could not obtain without them. But, thanks to these procedures, [...] numberless problems that would have remained isolated and would have required new efforts as they turn up are now arranged in an organic set.
>
> (Gauss 1975)

And again, in a letter to Schumacher:

> "It is the character of modern mathematics that through our language of signs and nomenclature, we possess a lever by which the most complicated arguments can be reduced to a particular mechanism. Science has thus gained an almost infinite richness, beauty, and solidity. But in the day-to-day use of this tool, science has lost almost as much as it has gained. How often is that lever applied only mechanically, although the authorization for it generally implies certain tacit hypotheses. I demand that in every use of a system of notation and in every use of a particular concept, each user be conscious of the original conditions and never regard as his property any products of the mechanism beyond its clear authorization."
>
> (Gauss 1975)

This paper surveys the evolution of structural dynamics from the second half of the nineteenth century to the first decades of the twentieth century in the light all of the foregoing statements, concentrating on those problems that benefited from interdisciplinary contact and which stimulated theoretical discussion.

HISTORICAL NOTES: THE CASE OF MODAL ANALYSIS

In order to offer a clear example of this interaction between the evolution of mathematical languages and structural dynamics, we shall focus our attention on the modal analysis of mechanical vibrations. If a student today takes a recently published textbook on this subject, he is likely to believe that it would not be possible to write the fundamental equations for the small oscillations of a N - degrees of freedom system and to look for their solution without the many elements of contemporary mathematics, which have been developed in the twentieth century as abstract algebraic N - dimensional spaces, vector analyses, the spectral theory of operators, functional analysis, etc.

As noted in previously, the solutions to these equations were attained by means of ingenious and elementary instruments, or, by peculiar expedients and laborious calcula.

We shall start from some preliminary studies about *De pendulis multifilibus,* published in Johann Bernoulli's *Opera omnia* (1742), and dwell to some extent on the great solution given by Lagrange in *Miscellanea Societatis Taurinensis* (1759-1760), and in his *Méchanique analytique* (1788).

The contributions offered by Claude-Louis Navier (1785-1836), Siméon-Dénis Poisson (1781-1840), Giovanni Antonio Amadeo Plana (1781–1865) and other minor authors will be examined and related to contemporary mathematical research (Fourier's series, 1822; Dirichlet's memoir, 1837). An almost unknown, but fundamental essay by Luigi Filippo Menabrea, Marchese di Valdora (1809-1896) will be presented in order to show that the main features of the modal analysis have been completely established by means of very elementary methods. Then Rayleigh's great paper of 1873 will be discussed. At the same time, a short outline of the mathematical revolution caused by the pioneering ideas of William Rowan Hamilton (1805-1865) (Hamilton 1843), Arthur Cayley (Caley 1843), Hermann Grassmann (Grassmann 1844), Josiah Willard Gibbs (1839-1903) (Gibbs 1881; Wilson 1947), Oliver Heaviside (1850-1925) (Heaviside 1883), August Föppl (1854-1924) (Föppl 1897), will help us understand the formal developments and improvements of modal analysis at the end of the nineteenth century and the first decades of the twentieth century. Of great importance was: the contributions of a great scholar such as Alexandre Mikaïlovitch Liapounov (1857-1918), another pioneer of geometrical methods in mathematical dynamics, especially his basic ideas on the development of stability problems; George Duffing (1861-1944), who studied mechanical devices in order to discover geometrical properties of dynamical systems with the theory of oscillations as his explicit goal; Jacques Hadamard (1865-1963) (Hadamard 1897), Tullio Levi-Civita (1873-1941), where mathematics lent to mechanics, and so on until the present routine formulation. Regarding this topic see also George David Birkhoff (1884-1944), Balthasar van der Pol (1889-1959), Nicholas Rashevsky (1899-1972), Chihiro Hayashi (1911-1986), etc.

A brief history of these subjects, indicating the most important contributors, includes: Louis Lagrange's analysis of the problems of small oscillations of discrete (elastic) systems, in a general case with general methods for N degrees of freedom. Then Navier, Poisson, Plana, etc improved Lagrange's solution. Menabrea's contribution, Rayleigh's fundamental memoir of (1873) and the treatise of Edward John Routh (1831-1907) (Routh 1877, 1898, 1920) were the basis for further development in the nineteenth century.

Rayleigh's *Theory of Sound* (1877) was the work, which "heralded the modern era of dynamics of elastic systems including, especially, engineering structures" (Charlton 1982, p. 163). 1897 was a crucial year: A. Kneser, A. M. Liapounov, J. Hadamard, T. Levi-Civita's contribution from 1896 to 1929, and Liapounov's essay of 1907 on the 'Problème général de la stabilité du mouvement' opened a new field for mathematical studies applied to structural mechanics and dynamics. New methods for the calculation of eigen-values and eigen-vectors, with application to structural dynamics, were developed in the first years of the twentieth century. The German school (E. Pohlhausen, Th. Poschl, E. Rausch, A. Tränkle, S. Gradstein, F.W. Waltking, K. Hohenmeser, R. Grammel, E. Fliegel, F. Reinitzhuber) produced in a few years (1921-1937) produced several interesting contributions to the development of mathematical problems.The problem related to the numerous degrees of freedom was resolved introducing integral equations (Vivanti 1916). The contributions of mathematicians, physicians & engineers like V. Volterra, D. Hilbert, H. Schmidt, Fr. Tricomi, A. Strassner, L. Collatz, G. Krall gave a great impulse to these studies and to the applications of structural mechanics (van den Dungen 1928).

THE APPLICATIONS OF STRUCTURAL DYNAMICS IN CIVIL ENGINEERING

The analysis of trusses and frames was examined and resolved by H. Reissner's fundamental papers on 'Schwingungsaufgaben aus der Theorie der Fachwerke' published in *Zeitschrift für Bauwesen* in 1899 and 1903. F. Bleich gave more contributions in his treatise to study iron bridges (Bleich 1924, pp. 41-77). Then F. Jodi, G. Krall, A. Galli applied this new formulation to several practical problems. Guido Alfani (1876-1940) produced important studies on the mechanical vibrations of buildings (Alfani 1909, 1910). A. Sommerfeld, A. Hertwig and H. Lorenz studied the problem of dynamic action on elastic soils (Love 1911, Krall 1940).

The applications related to the vibrations of machinery, engines, cars, and other similar topics were developed at the start of the twentieth century. I. Radinger, A. Stodola, I. Heun, H. Lorentz, W. Hort, R. v. Mises, and many others scientists have all solved practical problems in this respect. In these topics, for instance, the application of structural dynamics to electric locomotives were improved by E. Meissner, K.E. Muller, A. C. Couwenhoven, and A. Wichert, while dynamic vibration absorbers were studied by A. Föppl, and K. Klotter. Finally, the critical speed of a rotating shaft was resolved by A. Stodola, A. Föppl and others minor scholars (Timoshenko 1927). The applications in the field of hydrodynamics (Hadamard 1903; Cisotti 1921; Lamb 1924) and ship dynamics stemmed from the studies of W. Froude, A. Kriloff, H. Frahm, and J. Horn (Krall 1940).

SPECIAL TOPICS

Quasi-harmonic vibrations is contained in Floquet's theory (1883), while mathematical implementations and applications of this theory was developed by F. Klein, G. Hamel, O. Haupt, and others. J. Horn, G. Duffing, and G. Hamel studied non-linear oscillations of 1 degree of freedom. Vito Volterra (1860-1940), between 1912 and 1924, analysed the problem of hereditary actions (Volterra 1924). He obtained a mathematical solution with a continuo care during two decades. Finally, the theory of adiabatic invariance is the result of the studies of P. Eherenfest, J. M. Burgers, E. Fermi, T. Levi-Civita, and its application to problems of structural dynamics due to Giulio Krall (Krall 1940).

THE VIBRATIONS OF CONTINUOUS SYSTEMS: ROPES, BARS, BEAMS AND PLATES

The early theories of the eighteenth century are developed in the nineteenth century by, Navier (1823), Cauchy (1827) and Poisson (1833), to be completed by Clebsch (1862). The analysis of elastic bars under longitudinal impact is in reference to Young's theorems (Young 1807, pp. 46-50), Navier and Babinet's early studies (1823 and 1829), Phillips' memoir (1864), and de Saint-Venant's first fundamental contributions (1867-1868). A full contemporary clarification of the question was gave by J.V. Boussinesq (Boussinesq 1882) and by the de Saint-Venant's final review in his edition of Clebsch's treatise (Clebsch 1883). The problem of elastic beams when subjected to lateral impact refers to early studies of Eaton Hodgkinson (1789-1861), published in 1834-1836 in the *British Association Report* (Hodgkinson 1846), leading to the "Report of the Commissioners appointed to inquire into the application of iron to railway structures. On this subject see also "Appendix A" (1849) of the cited Report and the two important note of Cox's (Cox 1848, 1849).

De Saint-Venant formulated a new approach in the middle of the nineteenth century (1854, 1857, 1865). Fundamental is his Note to § 61 of Clebsch's treatise (Clebsch 1883). J. V. Boussinesq gave a solution in finite terms (Boussinesq 1885) and Alfred-Aimé Flamant (1839-1914) made some important remarks (Flamant 1886). Further interesting contributions are the book of Timoshenko (1932, pp. 41-47), and G. Krall (Krall 1940).

The Willis Problem or the "resilience" of a beam subject to a travelling load, born out of a question submitted to the above-mentioned Commission (1847). Its "Report-Appendix B" (Willis et al.) refers to Willis' experiments at Portsmouth and at Cambridge. George Gabriel Stokes (1819-1903) in 1849 gave an implementation of the solution, while a new approach of the same problem was formulated some years later by E. Phillips (1821-1889), then for the problem of the dynamical action of a moving load on a bridge (Phillips 1855). Moreover the latter also took up the forced longitudinal and lateral vibration of bars and gave solutions to such problems as that of the longitudinal vibration of a bar, one end of which is subjected to the action of a periodic force (Phillips 1864). The last important improvements of Phillips approach are de Saint-Venant's contribution of 1883 (de Saint-Venant 1883), the Boussinesq's solution (Boussinesq 1883) and the Resal's approach (Résal 1882). J. Melan, and H. Zimmermann later published further important contributions. Then, A. Kriloff defined an alternative approach. The debate continues with the contributions of F. Bleich, W. Prager, S. Timoshenko, and G. Krall (Krall 1940).

The most fundamental theory of transverse vibrations of a rod, with variable cross-section, started with Kirchhoff's basic memoir (Kirchhoff 1897). Some application of variations methods to this problem was given by L. Gümbel, J. Morrow; with reference to ship dynamics, by S. D. Taylor, O. Berling, A. Kriloff & K. E. Muller, J. J. Koch; with reference to machineries and engines by O. Schlick, A. Stodola, E. Schwerin, and many others authors (Stodola 1924).

Particular structures such as frames, continuous beams, rings, etc. and their problems were analysed by several authors: W. Kaufmann, K. Klotter, W. Prager, S. Timoshenko, K. Federhofer, W. Mudrak, A. Galli, G. Krall (Timoshenko 1932). Moreover, H. G. Küssner studied the effects of an axial load on transverse vibrations, and then L. Pochhammer resolved the problem of longitudinal and torsion vibrations. Finally, studies on plates was developed by S. Germain, G. R. Kirchhoff, F. Bernard, J. W. Rayleigh, H. Lamb & R. W. Southwell, K. Klotter, H. Reissner, H. Schmidt, R. Grammel, etc. These studies opened a new trend in structural analysis of plates and shells under dynamics loads (Love 1892-93, Timoshenko 1953, Timoshenko 1959).

FINAL REMARKS

This short account on the history of structural dynamics is obviously not complete, and it may appear as simply a listing of problems and authors. This is due to the large number of topics relevant to the subject, the numerous studies, problems, and application of case studies conducted over time. The complexity of these issues within a single aspect of mechanics, such as structural applied mechanics, stimulated many scientists to work and produce papers in a large number of mathematical domains. Therefore, it was not the ambition of this paper to cover all subject matter related to the history of dynamics, but only to offer the reader a first step that would introduce him to the interesting and complex matter of the history of mechanics and applied mechanics in the field of construction (Truesdell 1968).

ACKNOWLEDGMENTS

This paper is dedicated to the memory of Edoardo Benvenuto (1940-1998). His teaching is ever-present in my work.

LIST OF REFERENCES

Alembert, J. Le Rond d', 1744. *Traité de l'Équilibre et du Mouvement des Fluides. Pour servir de suite au Traité de Dynamique*; Paris: David.

Alembert, J. Le Rond d', 1758. *Traité de dynamique, dans lequel les loix de l'équilibre & du mouvement des corps sont réduits au plus petit nombre possible & démontrées d'une manière nouvelle, & où l'on donne un principe général pour trouver le mouvement de plusieurs corps qui agissent les uns sur les autres d'une manière quelconque. Par M. D'Alembert.* Paris: David.

Alfani, G., 1909. *I terremoti e le case: appunti popolari di sismologia*. Firenze: Alfani e Venturi.

Alfani, G., 1910. *Alcuni studi sulle vibrazioni meccaniche dei fabbricati*. Firenze: Alfani e Venturi.

Appell, P. É., 1904-1919. *Traité de mécanique rationnelle*. Paris: Gauthier-Villars.

Aquinas (Aquinayis), T., 1509. *Prima (et Secunda) Secunde (Summae) Sancti Thomae Aquinatis ordinis praedicatorum una cum annotationibus cotationibusque nuper per R. P. fratrem Matheum Sicculum eiusdem ordinis sancte theologiae magistrum additis*. Venetiis: impressa per Philippum Mantuanum expensis domini giuntini de Giunta Florentini.

Babinet, J., 1829. *Sur les Couleurs des réseaux, par M. Babinet, Lu à la Société philomatique, le 8 décembre 1827*. Paris: imprimerie de C. Thuau (see also: Babinet, J., 1855-68. *Etudes et lectures sur les sciences d'observation et leurs applications pratiques*. Paris: Mallet-Bachelier).

Béghin, H., 1921. *Statique et dynamique*. Paris: A. Colin.

Bernard, F., 1860. *Mémoire sur les vibrations des membranes élastiques*, "Comptes Rendus", vol. 51, pp. 322-325.

Bernoulli, Daniel, 1738. *Hidrodinamica, sive de Viribus et Motibus Fluidorum commentarii ...*, Argentorati: Johannis Reinholdi Dulseckeri.

Bernoulli Johann 1742. *Opera omnia, tam antea sparsim edita, quam hactenus inedita ... quibus continentur ea, quae ab A. 1690 usque ad A. 1727 prodierunt. Accedunt Lectiones Mathematicae de Calculo Integralium atque Anedokta*. Vol. I. Lausannae & Genevae (Genève et Lausanne): Bousquet.

Bleich, F., 1924. *Theorie und Berechnung der eisernen Brücken*. Berlin: J. Springer.

Born, M., 1954. *Dynamical theory of crystal lattices*, Oxford: Clarendon Press.

Boussinesq, J.V., 1882. "Sur la détermination de l'épaisseur minimum que doit avoir un mur vertical, d'une hauteur et d'une densité données, pour contenir un massif terreux, sans cohésion, dont la surface supérieure est horizontale", *Annales des Ponts et Chaussées*, 6ème série, I sem., vol. 3, pp. 625-643.

Boussinesq, J. V., 1883. "Du choc longitudinal d'une verre prismatique, fixée à un bout et heurtée à l'autre" *Comptes Rendus des Séances de l'Académie des Sciences*, vol. 97, pp. 154-157.

Boussinesq, J.V., 1885. *Applications des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques*. Paris: Gauthier-Villars.

Burali-Forti, C. and Boggio, T., 1921. *Meccanica razionale*. Torino-Genova: S. Lattes & Co.

Burgatti, P., 1921. *Lezioni di Meccanica razionale*. Bologna: Zanichelli.

Cauchy, A.-L., 1827a. "Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques", *Exercices de Mathématiques*, vol. 2, pp. 42-59.

Cauchy, A.-L., 1827b. "Sur la condensation et la dilatation des corps solides *Exercices de Mathématiques*, vol. 2 , pp. 60-69.

Cauchy, A.-L., 1827c. "Sur les relations qui existent, dans l'état d'équilibre d'un corps solide ou fluide, entre les pressions ou tensions et les forces accélératrices", *Exercices de Mathématiques*, vol. 2, pp. 108-111.

Cayley, A., 1843. "On the Intersection of Curves", *Cambridge Mathematical Journal*, vol. 3, pp. 211-213.

Cayley, A., 1843. "On the Motion of Rotation of a Solid Body", *Cambridge Mathematical Journal*, vol. 3, pp. 264-267.

Cayley, A., 1895. *An elementary treatise on elliptic functions*. London: George Bell and Sons.

Charlton, T.M., 1982. *A history of the theory of structures in the nineteenth century*. Cambridge: University Press.

Cisotti, U., 1921. *Idromeccanica piana*. Bologna: Zanichelli.

Clebsch, A., 1862. *Theorie der Elastizität Fester Körper*. Leipzig: B. G. Teubner.
Colonnetti, G., 1929. *Principii di dinamica. Seconda edizione riveduta ed accresciuta*. Torino: U.T.E.T.

Courant, R. and Hilbert, D., 1924. *Methoden der Mathematischen Physik*. Berlin: J. Springer.

Cox, H. 1848. "The dynamical deflexion and strain of railway girders", *Civil Engineer and Architect's Journal*, vol. 11, pp. 258-264.

Cox, H. 1849. “On impact on elastic beams”, *Cambridge Philosophical Transaction*, vol. 9, part. I, pp. 73-78.

Dirichlet (Lejeune-) P. G., 1837. “Sur l’usage des intégrales définies dans la sommation des séries finies ou infinies”, *Crelle, Journal für die reine und angewandte Mathematik,* vol. 17, s. 57-67, pp. 257-270.

Dugas, R., 1950. *Histoire de la Mécanique*. Neuchâtel: Ed. du Griffon.

Duhem, P., 1903. *L’évolution de la mécanique*. Paris: Joanin.

Euler, L., 1736. *Mechanica sive Motus Scientia Analytice exposita ...* . Petropoli: ex Typographia Academiae Scientiarum.

Flamant, A., 1886. *Stabilité des constructions, Résistance des matériaux*. Paris: Baudry et Cie.

Floquet, G., 1883. “Sur les équations différentielles linéaires à coefficients périodiques”, *Annales de l’École Normale Supérieure*, 2éme série, vol. 12, pp. 44-88.

Föppl, A., 1897. “Ziele und Methoden der technischen Mechanik”, *Deutsche Mathematiker Vereinigung*, vol. 6, pp. 99-110.

Föppl, A., 1931. *Grundzüge der technischen Schwingungslehre*. Berlin: J. Springer.

Föppl, A., 1901-10. *Vorlesungen über technische Mechanik, von Dr. Aug. Föppl*. Leipzig u. Berlin: B. G. Teubner.

Fourier, J., 1822. *Théorie analitique de la Chaleur*, Paris: chez Firmin Didot, Père et Fils.

Galilei, G., 1933. *Le opere di Galileo Galilei*. Firenze: Edizione Nazionale italiana.

Gauss, C. F., 1975. *Carl Friedrich Gauss Werke, Briefwechsel mit H. C. Schumacher. Teil 1, Teil 2 und Teil 3*. Hildesheim, New York: Georg Olms Verlag.

Geiger, J., 1927. *Mechanische Schwingungen und ihre Messung*. Berlin: J. Springer.

Gibbs, J. W. 1881. *Vector Analysis*. Privately printed notes on vector analysis for his students.

Grassmann, H. G., 1894. *Die Ausdehnungslehre von 1844 und die geometrische Analyse, unter der Mitwirkung von Eduard Studie herausgegeben von Friedrich Engel*. Leipzig: B. G. Teubner.

Hadamard, J., 1897. "Sur certaines propriétés des trajectoires en Dynamique", *Journal de Mathématiques Pures Appliquées ou Journal de Liouville*, V série, vol. 3, fasc. 4, pp. 331-388.

Hadamard, J., 1903. *Leçons sur la propagations des ondes et les équations de l'Hydrodynamique*. Paris: Hermann.

Hamilton, W.R., 1843. "On a new Species of Imaginary Quantities connected with a theory of Quaternions", *Proceedings of the Royal Irish Academy*, vol. 2, pp. 424-434.

Heaviside, O., 1894. *Electromagnetic theory. By Oliver Heaviside*. London: "The Electrician" printing and publishing company.

Hodgkinson, E., 1846. *Experimental Researches on the Strength and Other Properties of Cast Iron*. London: John Weale.

Hort, W., 1922. *Technische Schwingungslehre*. Berlin: J. Springer.

Huygens, C., 1673. *Horologium Oscillatorium. Sive de Motu Pendulorum ad Horologia aptato Demonstrationes Geometricae*. Paris: F. Muguet.

Kirchhoff, G. R., 1848. "Note relative à la théorie de l'équilibre et du mouvement d'une plaque élastique", *Comptes Rendus*, vol. 27, pp. 394-397.

Kirchhoff, G. R., 1849. "Note sur les vibrations d'une plaque circulaire", *Comptes Rendus hebdomadaires des Séances de l'Académie des Sciences*, vol. 29, pp. 753-756.

Kirchhoff, G. R., 1897. *Vorlesungen über mathematische Physik*. Vol. I: Mechanik. Leipzig: B.G. Teubner.

Klein, F., 1928. *Vorlesungen über Elementarmathematik*, vol. 3. Berlin: J. Springer.

Krall, G., 1940. *Meccanica tecnica delle vibrazioni*. Bologna: Zanichelli.

Lagrange, L., 1760. "Nouvelles recherches sur la nature et la propagation du son par M. De La Grange", *Miscellanea Taurinensia ou Mélanges de Philosophie et de Mathématique de la Société Royale de Turin Pour les Années 1760-1761*, Tomus alter, 2nd part, pp. 11-172.

Lagrange, L., 1788. *Méchanique analytique*. Paris: la Veuve Desaint.

Lamb, H. 1914. *Dynamics*. Cambridge: University Press.

Lamb, H. 1924. *Hydrodynamics*. Cambridge: University Press.

Lamb, H., 1925. *The dynamical theory of sound*. London: Arnold & C.

Laplace, P. S., 1799-1825. *Traité de mécanique céleste, par P. S. Laplace*. Paris: Duprat.

Lecornu, L.-F.-A., 1918. *Cours de mécanique*. Paris: Gauthier-Villars.

Lehr, E., 1930. *Schwingungstechnik*. Berlin: J. Springer.

Levi-Civita, T. and Amaldi, U., 1922, 1927. *Lezioni di Meccanica razionale*. Bologna: Zanichelli.

Liapounov, A.M., 1892. *Théorie générale de la stabilité*. Original Russian Dissertation.

Liapounov, A.M., 1907. "Problème général de la stabilité du mouvement", *Annales de la Faculté des Sciences de Toulouse*, 2éme série, vol. 9, pp. 203-474.

Lord Kelvin and Tait, P.G., 1912. *Treatise on natural philosophy*. Cambridge: University Press.

Lorenz, H., 1902. *Technische Mechanik*, vol. 1-4. München u. Berlin: von R. Oldenburg.

Love, A.E.H., 1911. *Some problems of Geodynamics*. Cambridge: University Press.

Love, A.E.H., 1921. *Theoretical Mechanics*. Cambridge: University Press.

Love, A.E.H., 1892-93. *A treatise on the mathematical theory of elasticity*. Cambridge: University Press.

Marcolongo, R. 1917-18. *Meccanica razionale*. Milano: Hoepli.

Navier, C. L., 1823. "Sur les lois de l'équilibre et du mouvement des corps solides élastiques", *Bulletin des Sciences de la Société Philomathique*, Série 2, vol. 2, pp. 177-181 (see also: Navier, C. L., 1827. "Sur les lois de l'équilibre et du mouvement des corps solides élastiques", *Mémoire de l'Académie Royale des Sciences*, vol. 7, pp. 375-393).

Newton, I., 1687. *Philosophiae naturalis principia mathematica*, Londini: jussu Societatis regiae ac typis Josephi Streater.

Phillips, E., 1855. "Calcul de la résistance des poutres droites telles que les ponts, les rails, etc., sous l'action d'une charge en mouvement", *Annales des Mines*, vol. 7, pp. 467-506.

Phillips, E., 1864. “Solution de divers problèmes de Mécanique, dans lesquels les conditions, sont des fonctions données du temps, et où l’on tient compte de l’inertie de toutes les parties du système”, *Journal de Mathématiques*, vol. 9, pp. 25-83.

Piola, G., 1845. “Intorno alle equazioni fondamentali del movimento di corpi qualsivogliano, considerati secondo la loro naturale forma e costituzione”, *Memorie di Matematica e Fisica della Società Italiana residente in Modena*, vol. 24, pp. 1-186.

Plana, G., 1815. “Mémoire sur les oscillations des lames élastiques”, *Journal de l'École Royale Polytechnique*, vol. 10, pp. 349-395.

Poincaré, H., 1890. “Sur le problème des trois corps et les équations de la Dynamique”, *Mémoire couronné du prix de S.M. le roi Oscar II de Suède. Acta Mathematica*, vol. 13, pp. 1-270.

Poincaré, H., 1905. *Leçons de mécanique céleste*. Paris: Gauthier-Villars.

Poisson, S. D., 1833. *Traité de Mécanique*. Paris: Bachelier.

Rayleigh, J. W. Strutt, Lord, 1873. “Some general theorems relating to vibrations”, *Proceedings of the London Mathematical Society,* vol. 4, pp. 357-368.

Rayleigh, J. W. Strutt, Lord, 1877-78. *The theory of sound*. London: Macmillan.

Résal, H., 1882. *Cours de Mécanique. Rédaction des élèves*. Paris: Ecole Polytechnique.

Routh, E.J., 1877. *A treatise on the Stability of a Given State of Motion*. London: Macmillan.

Routh, E.J., 1898. *A Treatise on the Dynamics of a Particle*. Cambridge: University Press.

Routh, E.J., 1920. *Treatise of the Dynamics of as System of Rigid Bodies*. London: Macmillan.

Saint-Venant, A.-J.-C. Barré de, 1854. “Solution du problème du choc transversal et de la résistance vive des barres élastiques appuyées aux extrémités”, *L’Institut*, vol. 22, pp. 61-63.

Saint-Venant, A.-J.-C. Barré de, 1857. “Mémoire sur l’impulsion transversale et la résistance vive des barres élastiques appuyées aux extrémités*”, Comptes Rendus de l’Académie des Sciences*, vol. 45, II sem., pp. 204-208

Saint-Venant, A.-J.-C. Barré de, 1865a. “Complément au Mémoire lu le 10 août 1857 sur l’impulsion transversale et la résistance vive des barres, verges ou poutres élastiques” (Extrait), *Comptes Rendus de l'Académie des Sciences*, vol. 60, I sem., pp. 42-47; vol. 61, I sem., pp. 33-37; vol. 62, I sem., pp. 130-134.

Saint-Venant, A.-J.-C. Barré de, 1865b. "Théorème nouveau de Mécanique, relatif aux forces vives vibratoires. Moyen pratique et élémentaire d'évaluer très approximativement, dans le plus grand nombre des cas, la flexion ou l'extension d'un système élastiques, due à un choc [Deuxième complément au Mémoire lu le 10 août 1857]", *Comptes Rendus de l'Académie des Sciences*, vol. 60, I sem., pp. 732-735.

Saint-Venant, A.-J.-C. Barré de, 1865c. "Troisième complément au Mémoire lu le 10 août 1857 sur l'impulsion et la résistance vive des pièces élastiques, et sur les forces vives dues aux mouvements vibratoires", *Comptes Rendus de l'Académie des Sciences*, vol. 61, II sem., pp. 33-37.

Saint-Venant, A.-J.-C. Barré de, 1867a. "Sur le choc longitudinal de deux barres parfaitement élastiques et sur la proportion de leur force vive qui est perdue par la translation ultérieure", *Société Philomathique de Paris*, n. 4, pp. 92-95.

Saint-Venant, A.-J.-C. Barré de, 1867b. "Sur le choc longitudinal de deux barres élastiques de grosseurs et matières semblables ou différentes, et sur la proportion de leur force vive qui est perdue pour la translation ultérieure; ... Et généralement sur le mouvement longitudinal d'un système de deux ou plusieurs prismes élastiques", *Journal de Mathématiques pures et appliquées de Liouville*, vol. 12, pp. 237-376.

Saint-Venant, A.-J.-C. Barré de, 1867c. "Démonstration élémentaire: (1°) de l'expression de la vitesse de propagation du son dans une barre élastique; (2°) des formules nouvelles données, dans une communication précédente, pour le choc longitudinal de deux barres, *Comptes Rendus de l'Académie des Sciences*, vol. 64, I sem., pp. 1192-1195.

Saint-Venant, A.-J.-C. Barré de, 1868. "Choc longitudinal de deux barres élastiques, dont l'une est extrêmement courte ou extrêmement roide par rapport à l'autre", *Comptes Rendus de l'Académie des Sciences*, vol. 66, I sem., pp. 650-653.

Saint-Venant, A.-J.-C. Barré de, 1883a. "Résistance vive ou dynamique des solides. Représentation graphique des lois du choc longitudinal, subi à une de ses extrémités par une tige ou barre prismatique assujettie à l'extrémité opposé, par MM. de Saint-Venant et Flamant", *Comptes Rendus de l'Académie des Sciences*, vol. 97, II sem., pp. 127-133; pp. 214-222; pp. 281-290; pp. 444-447.

Saint-Venant, A.-J.-C. Barré de, 1883b. *Détermination et représentation graphique des lois du choc longitudinal d'une tige ou barre élastique prismatique, par MM. de Saint-Venant et Flamant,* Paris: Gauthier-Villars.

Saint-Venant, A.-J.-C. Barré de, 1883c. *Théorie de l'élasticité des corps solides de Clebsch, traduite par MM. Barré de Saint-Venant et Flamant , avec notes étendues de M. de Saint-Venant*. Paris: Dunod.

Schneider, E., 1924. *Mathematische Schwingungslehre*. Berlin: J. Springer.

Stodola, A., 1924. *Dampf- und Gas-Turbinen*. Berlin: J. Springer.

Stokes, G. G., 1849. "On the dynamical theory of diffraction", *Cambridge philosophical Transactions*, vol. 9, pp. 1-62; 243-328.

Strassner, A., 1925. *Neuere Methoden zur Statik der Rahmentragwerke*. Berlin: W. Ernst.

Thomson, W. & Tait, P. G., 1867. *Treatise on Natural Philosophy*. Cambridge: University Press.

Timoshenko, S., 1927. *Method of Analysis of Statical and Dynamical Stresses in Rails*. Zurich: Orell Füssli.

Timoshenko, S., 1928. *Vibration problems in engineering*. New York: van Nostrand Co.

Timoshenko, S., 1932. *Schwingungsprobleme der Technik*. Berlin: Springer.

Timoshenko, S., 1953 . *History of (the) Strength of Materials*. New York: McGraw-Hill.

Timoshenko, S., 1959. *Theory of Plates and Shells*. New York: McGraw-Hill.

Todhunter, I. and Pearson, K., 1960. *A History of the Theory of Elasticity*. New York: Dover Publications Inc.

Truesdell, C. A., 1968. *Essays in the History of Mechanics*, Berlin-Heidelberg-New York: Springer-Verlag.

Van den Dungen, M.F.H., 1928. *Les problèmes généraux de la technique des vibrations*. Paris: Gauthier-Villars.

Vivanti, G., 1916. *Elementi della teoria delle equazioni integrali lineari*. Milano: U. Hoepli.

Volterra, V., 1924. *Saggi scientifici*. Bologna: Zanichelli.

Wilson, E. B., 1947. *The Early Work of Willard Gibbs in Applied Mechanics. A text Book for the Use of Students of Mathematics and Physics and Founded upon the Lectures of J. Willard Gibbs*. New Haven: Yale University Press.

Wittaker, E. T., 1917. *Analytical Dynamics*. Cambridge: University Press.

Young, T., 1807. *A Course of Lectures on Natural Philosophy and the Mechanical Arts*. London: J. Johnson (London: Taylor and Walton, 1845).