The Square Cube Law: from Vitruvius to Gaudí

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INTRODUCTION

The great majority of experts, professionals and students interested in historical construction, have only a certain notion of the square cube law in the best of cases, and very few are aware of its real consequences in specific cases, such as those which are derived from the expression “giants are frail”. Of course, this is only my personal opinion, but my experience has brought me to see it this way. Indeed, all this may lead to mistakes on a gigantic scale. An assessment of the work of the Catalan architect Antoni Gaudí provides some good examples in this respect.

Galileo was the first to enunciate the law (1638). His famous sketch of two bones, one short and slender, the other longer and much thicker, is a good representative icon of the rule; it is important to point out that bones must not only work in compression but also especially in tension due to the effects of the flexure to which they are normally subjected. An extensive explanatory exposition may be found in Huerta (2004, pp. 387-94).

Even so, I believe it would be well to give a brief explanation of the law together, precisely, with one of the few cases where Gaudí used a lintel working in flexure for the purpose of covering the top of an opening: the entrance doorway to the upper church of the Güell Colony (fig.1) which, as is generally known, he did not finish (González 2002c, pp. 170-80, www.razones-cripta-gaudi.com).

Figure 1a shows the trilith just as Gaudí left it. At the intrados of the lintel, due to the stone’s own weight, there is a tensile stress of about 0.2 N/mm². We do not know how this structure would have been loaded on top, but it would surely have been subjected to a greater stress. The lintel cannot count on the help of other contiguous lintels as happens in a Greek temple (Heyman 1972), so to avoid failure it must possess sufficient tensile strength in its intrados.

Figure 1b shows the first supposition of this explanation. By means of an infographic trick, the span of the opening has been doubled but not the lintel’s thickness. The calculation gives a tension of about 0.8 N/mm². A look at the experience of historical builders allows us to verify that they did not usually use a lintel of such proportions for such a span. Although on paper this stress does not appear excessive, everything indicates that it entails a risk of failure that one would not be willing to risk. Remember the picture of the broken architrave of the Temple of Zeus on the Acropolis in Athens published by Heyman in the aforementioned article. It is not a structural catastrophe but it could be a financial one for the builder.
A naïve conclusion: if one does not wish to increase the stress on doubling the span, it is necessary to double the thickness, in other words, it is necessary to maintain the same proportion (fig.1c). Nevertheless, the calculation holds that by doubling everything, the stress also doubles, rising to 0.4 N/mm².

Let us suppose that this stress is not admissible either, that is to say, that we cannot exceed the initial stress. On carrying out the respective calculation, one obtains a quadrupled thickness that will finally maintain the initial stress of 0.2 N/mm² (fig.1d).

![Figure 1. On the left, the original trilith in its real size (a). To its right, the first transformation (b), followed by the other two (c) and (d) (photograph by the author)](image)

This was what Galileo said: in the case of flexure, on increasing the size, if the tensile strength cannot be increased it is necessary to increase the thickness of the piece to twice the thickness that would result from maintaining the proportion. Hence, the bulky bone. Now, obviously, this is only so in the case where, on increasing the size, the tensile stress exceeds the strength of the material involved. If the strength of the material were extremely high, as it would be in steel bones, for example, this rule would not apply. Similar to this situation is that of the historical building subjected to compression: the compressive strength is usually very fair from the compressive stresses, although this is not always the case, of course.

As has been previously mentioned, there are many cases where it is found that historical builders, albeit unacquainted with the law as such in the case of flexure, were nevertheless acquainted with its effects and acted accordingly. The case that best demonstrates this assertion is that of the handling of the proportions of the classical orders.

**Vitruvius and the classical orders**

In his Book IV, Vitruvius, within a rigid code on the proportions between the various parts of a composition based on the orders, admits a very broad range for the intercolumniation, a variable that is determined by the span to be covered by the architrave. Assuming the diastyle (intercolumniation/modulus = 21/4) to be perfect, he still accepts the pycnostyle (i/m = 1 ½) for small spans and the aerostyle (i/m > 3) for large spans, although in the latter case stone does not offer a valid solution and timber is accepted. Some cases where, using a one-piece architrave, the
pycnostyle (that is, the intercolumniation of smallest span) was indispensable, were those of the large temples, the gigantic ones of Mars Ultor in Rome and of Jupiter in Baalbek \((i/m = 1.25)\).

Obviously, however, the practicality of the Romans led them to seek solutions that would not depend on the thickness of the piece, such as by unloading the architrave as far as possible by means of camouflaged arches (fig.2) or by using the flat arch, also called the *plate-bande* (fig.3). A “side effect” was the respective thrust, which was solved in both cases by means of abutments or tie bars.

Figure 2. Entablature of the Maison Carré in Nîmes. Highlighted in black are the joints between the pieces of the frieze, which form a flat arch unloading the monolithic architrave (photo. by the author)

Figure 3. Entablature of the Forum of Pompeii, in which the architrave and frieze are joined to form the voussoirs of a flat arch (photograph by the author)
If we now return to the square cube, Galileo could have said that, in addition to increasing greatly the thickness or increasing the strength of the material, the static diagram of a lintel or beam could be switched for that of a flat arch... although I cannot imagine how he could have applied this to an elephant bone!

The Renaissance encountered the same problem and its solutions were similar: either timber camouflaged beneath stucco, or flat arches with abutments or tie bars. The Baroque period and French Neoclassicism carried things to such an extreme in increasing size and intercolumniation, that it was necessary to implement exceptional solutions: Ste. Geneviève is a clear example (fig.4). In the nineteenth century, Viollet-le-Duc voiced a severe criticism in this respect (Viollet 1858-1868, also cited by Heyman 1972), to which Gaudí paid little attention shortly afterwards in the façade of La Pedrera (González 2002b).

It has previously been said that Gaudí was not very partial to straight elements working in flexure since his vital leitmotiv was self-balanced arches (Huerta 2003). If we search for some other example, in addition to the one just analysed of the Güell Colony, which may serve as a basis for our reflection on the square cube law, we find one in Gaudí’s youthful period at the Güell Palace in Barcelona. The solution that he used there, which was not inspired by the classical orders but indeed by the practical Roman solutions, was a new one in history.

Although this solution was presented at an earlier congress (González 2005), it would seem appropriate to return to it in greater detail in view of the broader attention to this event and the special interest of this little known aspect.
The *plate-bande* of the Güell Palace

The two volumes that flank the entrance vestibule of the building are marked by straight horizontal lower edges supported on free-standing pillars with a geometrical structure that may recall a work based on the use of piles and girders of reinforced concrete. Of course, in this case everything is made with ashlar and monolithic shafts of almost marmoreal limestone (fig.5).

![Figure 5. View from the entrance vestibule of the Güell Palace. In the rear of the picture, the parabolic arch gives access to the street. On the right, the volume on pillars (historical photograph)](image)

The structural method that was used is based on the properties of the stone alone, but it is very different from the structural method of the contiguous façade of parabolic arches: in this interior wall, flat arches are used. Upon these flat arches rest load-bearing walls that receive isolated loads from the upper floors. The perfect solution would have been a lintel of great thickness; another less costly solution would be that which was used in the peristyle of the Forum in Pompeii, for example. That was not the case here, however, where Gaudi designed an original capital that also acts as a double skewback (fig.6).

The balance of thrusts is achieved by the use of an iron tie bar embedded in the pavement of the floor assembly that rests on the arch; the tie bar is anchored in the capital at one end and in a pillar at somewhat of a distance (fig.7).
This method did not work well at all since it is the cause of one of the most notable lesions of the building, one that Gaudí should doubtless have envisaged. A detailed observation of the flat arches and the walls that rest on them, allows it to be seen, as it would have been easy to foresee, that the tie bars have not been able to immobilise completely the skewback capitals, so the arches have descended at their keystones. This has produced a distribution of pressure lines that converge on the rigid parts, that is to say, the pointed tips of the capitals, where intense increases of the compressive stresses have arisen. If the loads are not excessive and the configuration of the ashlar is perfectly homogeneous, these stresses are still bearable by the stone, which consists of a very compact limestone (fig.8).
Figure 8. Owing to the movement of the pillars, the keystones descend and the load becomes concentrated on the pointed tip of the capital, causing the stress to reach inadmissible values (González 2006a).

If any of the foregoing conditions are not met, however, or if none of them are met, the stress may reach the point of failure. Indeed, this is what happened in the wall on the left of the vestibule, which showed a spectacular crack that broke the wall from the pointed tip of the capital to the support point on the upper girder that transmits a very large load. The use of a flat arch working in compression as an alternative to a heavily loaded giant stone lintel working in flexure did not work well at all in this case (Casals 2004).

No other similar cases are found in Gaudí’s work in stone since he either used steel in the Güell Palace (although very little), and steel is a material to which the square cube law does not apply, or an incipient reinforced concrete at Güell Park, to which the law does not apply either. His work is based on a profusion of arches of small span that only support reasonable compressions. Nevertheless, there are two other cases where we may continue this study of the square cube law, but now precisely with respect to compressions.

As previously mentioned, if a material were of infinite strength, the square cube law would not apply to it. Accordingly, since the compressive strengths are very far removed, in principle, from the usual stresses in masonry buildings, the law does not affect them: the existence of giants that work in compression alone is possible. There is one case, however, that poses a certain doubt in this respect: the Sagrada Familia basilica in Barcelona.

**The Sagrada Familia giant**

In 1883 the young architect Antoni Gaudí was commissioned to continue the work on a Neo-Gothic church that was under construction and that had been previously designed by another architect. Neo-Classicism had already gone into crisis and it had been replaced by other “neos”, among which the Neo-Gothic achieved the greatest strength. Taking the Neo-Gothic as a basis, Viollet-le-Duc
became the ideologist of a new architecture that lent renewed protagonism to the coherence between form and material (González 1993, pp. 253-67). Additionally, a new structural modelling tool, graphic statics (Culmann 1866), had burst into the architecture schools and into professional activity, precisely to facilitate the design of the Neo-Gothic churches, among many other applications.

For Antoni Gaudí, however, rigorously following the essence of Viollet’s proposals, the Neo-Gothic was not the path to be followed. The balance of the thrusts produced by the Gothic groined vaulting required the use of buttresses and flying buttresses. These were elements that Gaudí would not admit. For him, Gothic art was imperfect and its stability was based on a permanent propping by flying buttresses, which he compared to the crutches used to support a crippled body (González 2002c, p. 68).

Gaudí’s first alternative proposal was ready after fifteen years, in 1898, in the first design of his own to replace one by the previous architect (fig.9). A large central parabolic arch which, with almost vertical thrusts, prevented the need for buttresses and flying buttresses, was combined with a still generally Neo-Gothic style. About twenty years more were to pass before the definitive solution would be established. During this interval, Gaudí succeeded in lending shape to his innovative ideas in his little great work: the Güell Colony church.

![Figure 9. Cross-sectional elevation of Gaudí’s design for the nave of the Sagrada Familia (González 2002c)](image-url)
As is widely known, this church’s general structural shape was antifunicular, that is, the shape obtained by inverting 180° the tensilely balanced form determined by a model of weighted strings. The specific shapes of the elements, structural or not, were based on another of Gaudí’s great innovations: ruled, hyperbolic paraboloidal, hyperboloidal and other surfaces (González 2002c, pp. 175-80, González 2003a). The definitive proposal established several years later for the Sagrada Familia was the direct consequence of the experience at the Güell Colony, although the string model was replaced by the use of graphic statics (Huerta 2003).

In both cases – that of the small church and that of the big one – the structural elements were solely linear, straight or curved, opposing the strictly axial stresses derived exclusively from the effect of gravity by means of their buckling stiffness and their compressive strength, whereby everything could be resolved with brick masonry in the small church, or with stone masonry in the big church. Mention was made in the previous Congress with respect to the paradoxes of Gaudí’s idea of surpassing the Gothic (González 2003a). What interests us here is its possible relation to the square cube law.

The relationship between the two churches is evident: only after the experience of the small church did Gaudí give final shape to the big one. Everything seems to indicate that, to some extent, there is a relation of proportionality between his main load-bearing elements: if the size of the small church is doubled, the similarity between the two churches would appear obvious (fig.10). We thus find ourselves with a case of an increase of size that maintains the proportion and that does not follow Galileo, of course.

Figure 10. Left, imaginary representation of Güell Colony church at real size. Centre, the same enlarged to coincide in height with nave of Sagrada Familia (González 2002c).
The square cube law remains present, however: according to Gaudí’s drawings (Puig Boada 1976), the load acting on the central pillar of the small church is 75 mT, while that which acts on the central pillar of the big church, according to his disciple Sugrañes (1923), is 1,100 mT., even though it should have been 75 x 8 = 600, according to the square cube law. The reason for this non-proportional increase lies in the change from brick to stone and the enormous dead weight placed on the roof to verticalise the thrusts of the central vaults, with the aim to avoid the use of the Gothic “crutches”. The compressive stresses reach up to 8 N/mm$^2$ (Sugrañes 1923). The graphic statics diagram (fig.11), however, allows us to see that, given the slenderness of the piece, any deviation of small magnitude, as there quite probably would be, can cause increases of up to 16 N/mm$^2$ or more as well as tensile stresses.

![Figure 11. Diagram by Sugrañes (1923) with the distribution pattern of the pressure lines placed on the cross-section of the central pillar of the design of the Sagrada Familia’s central nave](image)

The consequences of all this are explained by the two architects who are now in charge of its structural calculation:

The methods based on antifuniculars will be [were], as is known, the basis of his structural analysis. This procedure provides an acceptable level of security under vertical loads, but it stands below what is demandable under the effect of wind and seismic stresses. This fact, together with the appearance of new materials and the disappearance of many of the materials used at the turn of the century, means that any present-day consideration of the Sagrada Familia must involve the use of malleable materials such as reinforced concrete.

(Buxadé 2001, p. 734)
One of the two architects (Buxadé), in a lecture given at the Association of Architects of Catalonia (Co.A.C.) in November 2002, assured moreover that the pillars of the crossing that receive the loads of the drum, measuring 170 m in height, were not sufficient even if made of super-reinforced high-strength concrete. Gaudí had already considered the possibility of using iron pillars (Martinell 1951, p. 59). The solution that is now foreseen, according to the aforementioned lecture, is to lighten the non-bearable load of the drum by means of light materials, such as titanium, for example.

It does not seem necessary to provide any further arguments to see that Gaudí was not consistent with respect to the square cube law. Were it to be made of stone, the Sagrada Familia giant should have had pillars that are much more similar to Galileo’s bulky bone. Not only is the compressive strength of stone masonry not infinite, but moreover the horizontal actions, which become larger the taller the building grows, may deviate the thrust lines and cause still more non-bearable compressive stresses or even tensile stresses. Since Gaudí maintained the proportions of his small model, there has been nothing else to do but to provide greater tensile strength with the steel of the reinforced concrete, something similar to what happened in the Baroque and French Neo-Classicism, or else, in accordance with Gaudí himself, to increase the compressive strength... with iron pillars! “Galileo” is inexorable. That is, the bulky bone becomes a bone of normal appearance but made of iron. In the following case, Gaudí is not to be blamed in the least, but rather the architects of the last half of the twentieth century are at fault.

**New York’s gigantic giant**

If ones reviews the daily press from the middle of January 2003, including both Spanish and international publications and especially those of New York City (www.hotelattraction.com/), it may be seen how much interest was aroused by the presentation of a virtual re-creation of the design for a 360 m tall hotel attributed to Gaudí (fig.12). The purpose was to compete for the occupation of the Ground Zero area left by the Twin Towers in 2001.

![Figure 12. A picture given out in January 2003 with a very fanciful view of the so-called Gaudí hotel set on the profile of New York City (www.hotelattraction.com/)](image-url)
The director of the Royal Gaudí Chair included in his grand book on the architect’s work (Bassegoda 1989) the complete report in which the son of one of Gaudí’s assistants, Jaume Matamala, possessing hardly any knowledge of construction, made public in 1956 a fact unknown to any of the architect’s biographers: the visit that some New York financial magnates had paid to Gaudí in 1908 to commission him for the building of a “mega-hotel”. The report from 1956 was accompanied by drawings, some attributed to Gaudí and others based on them. All this is generally known and, in addition to the aforementioned book, another publication that appeared in the flurry of 2002 (Montaner 2003) provides further information on this matter.

A quick look at one of the sketches attributable (but not attributed) to Gaudí (fig.13) allows it to be deduced that the building was conceived under the same criteria as those applied by the architect to devise the hollow towers of his buildings – those of the Tangiers missions, of the Sagrada Familia and of the Güell Colony church. The pattern is the one that has been previously mentioned: if the shape adopted by a chain supported by its two ends, that is to say, a catenary, is inverted 180º, a figure is obtained that is adopted as the directrix of a balanced stone masonry arch. The same criterion may be applied to a hollow dome of revolution generated by rotating the arch round its vertical axis, just as Gaudí did in his slender towers which could be built of masonry without fearing any adverse effect.

Figure 13. This is one sketch of the few attributable (but not attributed) to Gaudí. The graphic scale on the left may perfectly well indicate that the building does not exceed 100 m (Matamala 1956)
Indeed, Gaudí was so proud of his method that he instructed his disciple Jeroni Martorell so that, in a hypothetical discussion with the French architects on the latter’s visit to Paris on the occasion of the exhibition organised by Güell, he could present this profile of Gaudi’s towers as a demonstration of having surpassed (exclusively from the formal standpoint, I would add) the “telescopic” towers of the Gothic (once again his obsession with surpassing the Gothic) (González 2002c, González 2003a). The argument of the inverted catenary shape is even that which Bassegoda advances to assure Gaudí’s authorship. If this is the shape of the profile, however, it is also reasonable to assume that the structure of the hotel would have to be of masonry, the only structure for which this shape has any meaning.

It would seem evident, just as I have previously stated on two occasions (González 2003c, González 2006b), that there are many things that do not fit in here: neither the historical moment (the world’s tallest skyscraper in 1908 was New York’s “Venetian” Metropolitan Life Insurance Tower, at 213 m), the building system (obviously, it should have been a metal structure), nor, above all, the metric references of the sketch (the hotel, or whatever it was, did not surpass 100 m). What I think it is important to add now to this reasoning, however, are the aspects relating, of course, to the square cube law. The extraordinarily surprising thing, at least for me, is that, since 1956, no one has made the most obvious objection of all, the one derived from Galileo’s law, which is perfectly applicable to this case: the purported hotel of 360 m in height is a gigantic version of the towers of the Sagrada Familia which are made of stone masonry.

Now, one thing is a hollow dome or tower, which is what those of the Sagrada Familia are structurally, or what the towers of the Tangiers missions or those of the Güell Colony church would have been, and another thing is a gigantic hotel with a height of 360 m in which there are dozens of horizontal elements that lend support to the floors of rooms, halls, etc. If we cast a glance at the works of Gaudí, the only case where a similar element is found is the level supported by arches that separates the two superimposed churches of the Güell Colony. In this case, the general profile is not that of a simple catenary but rather the shape determined by the thrusts of the arches that give the exterior wall a greater slope the nearer it is to its base.

None of this is to be observed in the case of the hotel, but rather the profile becomes more vertical the nearer it is to the ground. This could only be achieved by reducing the lateral thrusts by means of arches of enormous rise, just as may be observed in the aforementioned sketch, and providing a hollow interior thanks to the pillars that receive these arches. Despite it all, however, on the upper levels may be seen some horizontal elements that could only have been floor assemblies based on steel girders and beams, whereby the inverted catenary shape would cease to have any meaning for even more reasons than those already mentioned.

Nevertheless, let us continue with the square cube law. Let us assume, in order to have some measurable reference, that the structure of the hotel would be similar to that seen earlier in the nave
of the Sagrada Familia, although the latter is somewhat denser, its total height including the roof being about 75 m. The proportion of the increase in size (360/75) would be between 4 and 5. Let us adopt the most favourable value, 4. If we were to maintain the proportion between the elements, the compressive stress at the base of the pillars would reach $8 \text{ N/mm}^2 \times 4 = 32 \text{ N/mm}^2$. That is to say, only 24. Obviously, this leaves much to be desired as a precise calculation, but I think it is more than sufficient as an order of magnitude. No masonry structure has ever reached these stresses or come anywhere near them. The solution would have required, of course, an increase of the thickness of all the load-bearing elements by a factor of at least 4, or perhaps of just 3, in other words, by following Galileo.

Now is the time to consider the structure of the only skyscraper built of masonry, the Monadnock Building in Chicago (1891), which stands 17 floors (64 m) tall. Its ground-floor walls are over 2.10 m thick, which means that at this level, the area occupied by the structural walls is equivalent to about 20% of the area of the plan. It was the first and last skyscraper in which this building method was used. What thickness would have been required by a skyscraper rising 360 m in height? 6 m, 7 m, 10 m? What percentage of the plan would its walls occupy? These are evidently meaningless questions, since it is meaningless to think of a 360 m skyscraper made of masonry. One may continue to adduce that it could have had a metal structure, but in that case, what would be the sense of the inverted catenary shape? Gaudi remained loyal to the coherent relationship between form and material, especially in the final period of his life. It would have made no sense to design a building of metal structure with a shape appropriate for stone. Gaudi knew perfectly well the difference between one and the other; he had used steel even in his early work at the Güell Palace. If he had adopted steel for the mega-hotel, its shape would have doubtless been extraordinarily original, but it would have been consistent with this material.

But regardless of whether or not Gaudi was the author of this mega-project, the most important thing of all, what is really surprising, is that, despite all that has been written about his work, including the mega-hotel (at the Library of the Association of Architects of Catalonia there are 777 bibliographic references to Gaudi and his work; see also González (2003b), no one, neither architect nor engineer, questioned this whole story by means of these or similar ideas between 1956 and 2003. As if this were not enough, the mega-hotel project was submitted with the acquiescence of the Government of Catalonia to nothing less than the competition to replace the Twin Towers in New York. Didn’t anyone know that the inverted catenary shape is only reasonable for masonry? Above all, didn’t anyone know that the limitation derived from the square cube law may come to hold true even for compressions alone? Did everyone doubt Gaudi’s structural coherence? If that was indeed the case, no one said so.

Of course, if the historical forms of domes, vaults or large bridges are maintained, as has been thoroughly demonstrated in Huerta 2004 (pp. 390-407), the compressive stresses do not surpass reasonable values. What happened with Gaudi, however, is that he came into a great contradiction:
he remained loyal till the end of his life to a historical material, masonry, but he broke radically with the forms to which this same material had given rise over the course of the centuries. When not only the form changed radically but the size of the work increased inordinately as well, the inevitable result was the super-reinforced concrete of the Sagrada Familia which, despite it all, requires a lightening of its loads through the replacement of stone by titanium.

There is absolutely no reason to blame Gaudí for the gigantic New York madness (if the architect did anything in this respect, it was only the sketch of a building of 100 m in height). The blame should go, rather, to all those who, without the slightest critical attitude based on a knowledge and understanding of historical construction, encouraged, between 1956 and no less than 2003 in New York, the unbridled fantasy of a possibly well-intentioned but scarcely enlightened dreamer. As was mentioned at the beginning, it was unquestionably a gigantic mistake.

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